COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **26** (1) (2006) 65–82

FLATTENING ANTICHAINS

ÁKOS KISVÖLCSEY

Received January 11, 2003

A flat antichain is a collection of incomparable subsets of a finite ground set, such that $|B|-|C|\le 1$ for every two members B,C. Using Lieby's results, we prove the Flat Antichain Conjecture, which says that for any antichain there exists a flat antichain having the same cardinality and average set size.

1. Introduction

In this paper we investigate subsets of the underlying set $[n]=\{1,2,\dots,n\}$. A family $\mathcal{A}\subseteq 2^{[n]}$ is called an *antichain* if for all distinct members $B,C\in\mathcal{A}$ we have $B\not\subset C$. The size of $\mathcal{A}$ is $|\mathcal{A}|$, the volume is $\mathrm{vol}(\mathcal{A})=\sum_{A\in\mathcal{A}}|A|$, and the average set size is $\mathrm{av}(\mathcal{A})=\frac{1}{|\mathcal{A}|}\sum_{A\in\mathcal{A}}|A|$. We say that $\mathcal{A}$ is *flat* if for all $A\in\mathcal{A}$ we have $|A|=d$ or $|A|=d+1$ for some non-negative integer d. A family $\mathcal{B}$ is a *flat counterpart* of $\mathcal{A}$ if $\mathcal{B}$ is flat, $|\mathcal{A}|=|\mathcal{B}|$ and $\mathrm{av}(\mathcal{A})=\mathrm{av}(\mathcal{B})$.

A *completely separating system* (CSS) is a family $\mathcal{C}$ of subsets of $[m]$ such that for each ordered pair (a,b), $a\neq b$, there is a set in $\mathcal{C}$ which contain a but does not contain b. It is a natural question to determine the minimum size of a k-uniform CSS: Ramsay, Roberts and Ruskey [9,10] have found upper and lower bounds on it.

The *dual* of a set system $\mathcal{A}=\{A_1,\dots,A_m\}\subseteq 2^{[n]}$ is the collection $\mathcal{A}^*=\{C_1,\dots,C_n\}\subseteq 2^{[m]}$, where $C_i=\{j : i\in A_j\}$. It is easy to see that $\mathcal{A}$ is an antichain if and only if $\mathcal{A}^*$ is a CSS. Since $\mathrm{vol}(\mathcal{A})=\mathrm{vol}(\mathcal{A}^*)$, a necessary condition for $\mathcal{C}\subseteq 2^{[m]}$ being a k-uniform CSS of size n is that there exists

Mathematics Subject Classification (2000): 05D05 an antichian of size m with volume kn, i.e. with average set size kn/m. Investigating this problem, Lieby [6] conjectured the following.

Flat Antichain Conjecture. *If $\mathcal{A}$ is an antichain, then there exists a flat antichain with the same size and average set size.*

Thus this conjecture would make easier to check whether a CSS exists with given parameters. However, this is also a nice problem itself.

The conjecture has been verified in several special cases. A result of Kleitman and Milner [5] implies that if $\mathcal{A}$ is an antichain with integral average set size, then the conjecture is true. Roberts [11] has solved the FAC for antichains with average set size at most 3. In her PhD thesis, Lieby has proven the conjecture if $\mathcal{A}$ is contained in 3 or 4 consecutive levels.

Theorem 1 (Lieby [7,8]). *Let $\mathcal{A}$ be an antichain, such that $|B|-|C|\leq 3$ for all $B, C\in\mathcal{A}$. Then the Flat Antichain Conjecture holds for $\mathcal{A}$.*

Brankovic, Lieby and Miller [1] have shown a weaker version of the conjecture, that is, there is a flat antichain with the same volume but not necessary the same size.

In the present paper we prove the FAC using Theorem 1 and a sufficient condition for the existence of an antichain on two levels.

2. Tools

Let $\mathcal{H}\subseteq\binom{[n]}{h}$ be a family of h-element sets; the *shadow* of $\mathcal{H}$ is defined as $\Delta\mathcal{H}=\{G:|G|=h-1, G\subset H\in\mathcal{H}\}$. Similarly, the *shade* of $\mathcal{H}$ is $\nabla\mathcal{H}=\{G:|G|=h+1, G\supset H\in\mathcal{H}\}$. By simple double counting argument, Sperner has obtained lower estimations on the shadow and the shade.

Lemma 1 (Sperner [12]). *Let $\mathcal{H}$ be a collection of h-element subsets of $[n]$. Then*

$$|\Delta\mathcal{H}|\geq\frac{h}{n-h+1}|\mathcal{H}|$$

$$|\nabla\mathcal{H}|\geq\frac{n-h}{h+1}|\mathcal{H}|.$$

For $0\leq i\leq n$, we denote by $\mathcal{A}_i$ the collection of i-element sets in $\mathcal{A}$, i.e. $\mathcal{A}_i=\{A\in\mathcal{A}:|A|=i\}$, and let $a_i=|\mathcal{A}_i|$. The vector $(a_0,\dots,a_n)$ is called the *profile* of $\mathcal{A}$.

Given two sets $A, B\subseteq[n]$, we say that A is smaller than B in the *squashed order* if the largest element of the symmetric difference of A and B is in B.

A *squashed antichain* is an antichain $\mathcal{A}$ if for all i the i-element subsets contained in a member of $\cup_{j\geq i}\mathcal{A}_j$ constitute an initial segment of the i-element sets in the squashed order. An important theorem about antichains and squashed antichains is the following.

Theorem 2 **(Clements [2], Daykin, et al. [3]).** *For each antichain, there exists a squashed antichain with the same profile.*

For fixed k, l let us take the pairs $(x,y)\in\mathbb{Z}^2$, $0\leq x\leq\binom{n}{k}$, $0\leq y\leq\binom{n}{l}$ for which there is no antichain $\mathcal{A}=\mathcal{A}_k\cup\mathcal{A}_l$ such that $|\mathcal{A}_k|=x$, $|\mathcal{A}_l|=y$. In [4] we have determined the convex hull of such pairs by giving its extreme points. So every point (x,y), $x\leq\binom{n}{k}$, $y\leq\binom{n}{l}$ are non-negative integers that lies outside of this convex set can be associated with an antichain on levels k, l. In particular, if the non-negative integers x, y satisfy the inequality $ux+vy<c$, where $u,v>0$ and c are appropriate constants then there is an antichain $\mathcal{A}=\mathcal{A}_k\cup\mathcal{A}_l$ with $|\mathcal{A}_k|=x$, $|\mathcal{A}_l|=y$ (see Fig. 1).

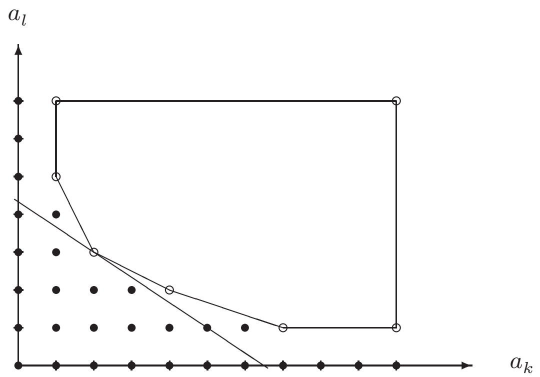

Fig. 1.

In order to obtain the best possible c for given u, v, we have computed the minimum of $ux+vy$ where (x,y) is in the convex hull. We have then established a theorem which has the special case for consecutive levels.

Theorem 3 **([4]).** *Let $u,v>0$ be given numbers, $\frac{1}{d}\leq\frac{u}{v}\leq n-d-1$ and let*

$$\gamma_d(u,v)=u\cdot\left(\sum_{j=2}^{n-d}\binom{\alpha_j}{j}+1\right)+v\cdot\left(\sum_{j=2}^{d+1}\binom{\beta_j}{j}+1\right),$$

where

$$\alpha_j = \min\Big\{\lceil(\tfrac{v}{u}+1)(j-1)\rceil - 1\,,\; d+j-1\Big\},$$

$$\beta_j = \min\Big\{\lfloor(\tfrac{u}{v}+1)(j-1)\rfloor,\; n-d+j-2\Big\}.$$

If for non-negative integers x, y we have

$$ux + vy < \gamma_d(u, v),$$

then there is an antichain $\mathcal{A}$ on levels $d, d{+}1$, such that $|\mathcal{A}_d| = x$, $|\mathcal{A}_{d+1}| = y$.

In the remaining part of this chapter, we give more explicit formulas on $\gamma_d(u,v)$ for some u, v.

Lemma 2. *Let $2d+1 \le n$. Then*

$$\gamma_d(1,1) = \binom{n}{d} - \binom{2d+1}{d} + \sum_{j=1}^{d}\left(\binom{2j-1}{j+1} + \binom{2j}{j+1}\right) + 2.$$

Proof. From Theorem 3 we have

$$\gamma_d(1,1) = \sum_{j=2}^{n-d}\binom{\min\{\,2(j-1)-1\,,\;d+j-1\,\}}{j} + 1$$
$$+ \sum_{j=2}^{d+1}\binom{\min\{\,2(j-1)\,,\;n-d+j-2\,\}}{j} + 1.$$

An easy calculation shows now the statement. ∎

Lemma 3. *If $n \le 3d+2$ then*

(1)
$$\gamma_d(2,1) = 2\sum_{j=2}^{n-d}\binom{\left\lceil\frac{3}{2}(j-1)\right\rceil - 1}{j} + \sum_{j=2}^{\left\lfloor\frac{n-d+1}{2}\right\rfloor}\binom{3(j-1)}{j}$$
$$+ \binom{n}{d+1} - \binom{\left\lfloor\frac{3}{2}(n-d-1)\right\rfloor + 1}{n-d-1} + 3,$$

while for $n \ge 3d+1$, we have

(2)
$$\gamma_d(2,1) = 2\sum_{j=2}^{2d+2}\binom{\left\lceil\frac{3}{2}(j-1)\right\rceil - 1}{j} + 2\binom{n}{d} - 2\binom{3d+2}{2d+2}$$
$$+ \sum_{j=2}^{d+1}\binom{3(j-1)}{j} + 3.$$

Proof. First, note that for $n=3d+1, 3d+2$, these formulas are equal. Using the notations of Theorem 3,

$$\alpha_j = \begin{cases} \left\lceil \frac{3}{2}(j-1) \right\rceil - 1 & \text{if } j \leq 2d+2 \\ d+j-1 & \text{otherwise,} \end{cases}$$

and

$$\beta_j = \begin{cases} 3(j-1) & \text{if } j \leq \frac{n-d+1}{2} \\ n-d+j-2 & \text{otherwise,} \end{cases}$$

and straightforward computations show the statement. ∎

Lemma 4. *If* $2d+1 \leq n$ *then*

$$\gamma_d(n-d-1, d) = d\binom{n-1}{d+1} + d\sum_{i=1}^{d-1}\binom{\left\lfloor \frac{d-i}{d}(n-1) \right\rfloor}{d-i+1} + d$$
$$+ (n-d-1)\binom{n-1}{d-1} - (n-d-1)\sum_{i=1}^{d-1}\binom{\left\lfloor \frac{d-i}{d}(n-1) \right\rfloor}{d-i}.$$

Proof. We use the notations of Theorem 3, again. Here

$$\alpha_j = \left\lceil \frac{n-1}{n-d-1}(j-1) \right\rceil - 1,$$

because

$$\frac{n-1}{n-d-1}(j-1) \leq d+j,$$

since $j \leq n-d$. Let us write α_j in the form $\alpha_j = d+j-i-1$, where

$$i = d+1-\left\lceil \frac{d}{n-d-1}(j-1) \right\rceil, \tag{3}$$

which is equivalent to

$$\frac{d-i}{d}(n-d-1)+1 < j \leq \frac{d-i+1}{d}(n-d-1)+1.$$

(3) with $2 \leq j \leq n-d$ and $2d+1 \leq n$ yield $1 \leq i \leq d$. We also observe that $2d+1 \leq n$ implies $\frac{d-i+1}{d}(n-d-1) - \frac{d-i}{d}(n-d-1) \geq 1$, so we have non-empty

sums below. We obtain

$$\sum_{j=2}^{n-d}\binom{\alpha_j}{j}=\sum_{i=1}^{d}\sum_{j=\left\lfloor\frac{d-i}{d}(n-d-1)\right\rfloor+2}^{\left\lfloor\frac{d-i+1}{d}(n-d-1)\right\rfloor+1}\binom{d+j-i-1}{j}$$
$$=\sum_{i=1}^{d}\left(\binom{\left\lfloor\frac{d-i+1}{d}(n-1)\right\rfloor}{d-i}-\binom{\left\lfloor\frac{d-i}{d}(n-1)\right\rfloor+1}{d-i}\right)$$
$$=\binom{n-1}{d-1}-\sum_{i=1}^{d-1}\left(\binom{\left\lfloor\frac{d-i}{d}(n-1)\right\rfloor+1}{d-i}-\binom{\left\lfloor\frac{d-i}{d}(n-1)\right\rfloor}{d-i-1}\right)-\binom{1}{0}$$
$$=\binom{n-1}{d-1}-\sum_{i=1}^{d-1}\binom{\left\lfloor\frac{d-i}{d}(n-1)\right\rfloor}{d-i}-1.$$

On the other hand,

$$\beta_j=\left\lfloor\frac{n-1}{d}(j-1)\right\rfloor,$$

because

$$\frac{n-1}{d}(j-1)\le n-d+j-2,$$

since $j\le d+1$. ∎

3. Proof of the Flat Antichain Conjecture

From now on, let $d\le \mathrm{av}(\mathcal{A})<d+1$ for some integer d. The main idea of the proof is the following. We find a weight function $w:\mathcal{A}\to\mathbb{R}$ such that

i) w is constant on the levels, i.e. $w(B)=w(C)=w_i$, for all $B,C\in\mathcal{A}$, $|B|=|C|=i$,
ii) w is linear: $w_i=si+t$,
iii) w_d, w_{d+1} are positive and $\frac{1}{d}\le\frac{w_d}{w_{d+1}}\le n-d-1$.

If for the total weight of $\mathcal{A}$, $w(\mathcal{A})=\sum_{i=0}^{n} w_i a_i$ we have

$$w(\mathcal{A})<\gamma_d(w_d,w_{d+1}),$$

then the conjecture is true for $\mathcal{A}$. Indeed, let x,y be the non-negative integers

$$x=(d+1)|\mathcal{A}|-\mathrm{vol}(\mathcal{A}),$$
$$y=\mathrm{vol}(\mathcal{A})-d|\mathcal{A}|.$$

Since w is linear, it is easy to see that $w_d x + w_{d+1} y = w(\mathcal{A})$, so Theorem 3 ensures the existence of an antichain $\mathcal{B}$ on levels $d, d+1$ with $|\mathcal{B}_d| = x$, $|\mathcal{B}_{d+1}| = y$. But

$$x + y = |\mathcal{A}|,$$

$$dx + (d+1)y = \mathrm{vol}(\mathcal{A}),$$

hence $\mathcal{B}$ has the same size and average set size as $\mathcal{A}$.

If the average set size of an antichain is $n/2$ for even n, then the conjecture is valid by Sperner's theorem [12]. In every other case we can assume that $2d+1 \le n$; otherwise, we turn the whole poset $2^{[n]}$ upside down, that is, we take the antichain $\mathcal{A}^c = \{[n] - A : A \in \mathcal{A}\}$, which has average set size $\mathrm{av}(\mathcal{A}^c) = n - \mathrm{av}(\mathcal{A})$. If an antichain $\mathcal{B}$ is a flat counterpart of $\mathcal{A}^c$, then $\mathcal{B}^c$ is clearly a flat antichain of size $|\mathcal{B}^c| = |\mathcal{A}|$ and average set size $\mathrm{av}(\mathcal{B}^c) = \mathrm{av}(\mathcal{A})$. Furthermore, we suppose $d > 0$, since otherwise $\mathrm{av}(\mathcal{A}) < 1$, so $\emptyset \in \mathcal{A}$ thus $\mathcal{A} = \{\emptyset\}$, and there is nothing to prove. Similarly, $d < n-1$.

In order to prove the FAC, we apply induction on $[n]$. For $n \le 3$, every antichain on $[n]$ is flat. If $n = 4$, then it is easy to see that, up to the permutation of the elements, there is only one antichain on $[n]$ which is not flat: $\{\{1,2,3\},\{4\}\}$. In this case, $\{\{1,2\},\{3,4\}\}$ shows the statement of the conjecture.

By Theorem 2, we can assume that $\mathcal{A}$ is a squashed antichain. Let $\mathcal{A}(\bar{n}) = \{A \in \mathcal{A} : n \notin A\}$ and $\mathcal{A}(n) = \{A \in \mathcal{A} : n \in A\}$ with average set sizes $k \le \mathrm{av}(\mathcal{A}(\bar{n})) < k+1$ and $l - 1 < \mathrm{av}(\mathcal{A}(n)) \le l$. Since $\mathcal{A}$ is squashed, we have $\min\{|B| : B \in \mathcal{A}(\bar{n})\} \ge \max\{|C| : C \in \mathcal{A}(n)\}$, thus $k \ge l$. By the induction hypothesis, there are antichains $\mathcal{B}$ and $\mathcal{B}'$ on $[n-1]$ that are flat counterparts of $\mathcal{A}(\bar{n})$ and $\{A - \{n\} : A \in \mathcal{A}(n)\}$, respectively. So the antichain $\mathcal{B}'' = \{B \cup \{n\} : B \in \mathcal{B}'\}$ is a flat counterpart of $\mathcal{A}(n)$.

Consequently, it is enough to investigate antichains for which $\mathcal{A} = \mathcal{A}_{l-1} \cup \mathcal{A}_l \cup \mathcal{A}_k \cup \mathcal{A}_{k+1}$ with $a_l, a_k > 0$, $l \le k$ and no member of $\mathcal{A}_k \cup \mathcal{A}_{k+1}$ contains n, but every member of $\mathcal{A}_{l-1} \cup \mathcal{A}_l$ contains n, thus $l > 0$, $k < n$. Moreover, if $\mathcal{A}_{l-1}$ is non-empty, then $l > 1$, while $\mathcal{A}_{k+1} \ne \emptyset$ implies $k < n-1$. In view of Theorem 1, we will suppose $k \ge l+2$. It is clear, that also $l - 1 \le d \le k$ holds.

Introduce the notations $\nabla a_{l-1} = |\nabla \mathcal{A}_{l-1}|$ and $\Delta a_{k+1} = |\Delta \mathcal{A}_{k+1}|$. Since $n \in A$ for all $A \in \mathcal{A}_{l-1}$, applying Lemma 1 for $\mathcal{A}_{l-1}$, $l > 1$ (with $n-1$ and $h = l-2$) and for $\mathcal{A}_{k+1}$, $k < n-1$ (with $n-1$ and $h = k+1$), we obtain

$$\nabla a_{l-1} \ge \frac{n-l+1}{l-1} a_{l-1}, \qquad \Delta a_{k+1} \ge \frac{k+1}{n-k-1} a_{k+1}. \tag{4}$$

$\mathcal{A}$ is an antichain, hence $\nabla \mathcal{A}_{l-1}, \mathcal{A}_l$ are disjoint sets, and $n \in A$ for all $A \in \nabla \mathcal{A}_{l-1} \cup \mathcal{A}_l$. Similarly, $\Delta \mathcal{A}_{k+1} \cup \mathcal{A}_k \subseteq \binom{[n-1]}{k}$, where $\Delta \mathcal{A}_{k+1}, \mathcal{A}_k$ are disjoint.

Therefore,

$$\nabla a_{l-1} + a_l \le \binom{n-1}{l-1}, \qquad a_k + \Delta a_{k+1} \le \binom{n-1}{k}. \tag{5}$$

We will separate four cases.

3.1. Case 1. $k \ge d+3,\ l = d+1$

First, we study the case when $n = 2d+1$. Let the weights be $w_j = 1$ for all $j = 0, \ldots, n$. By (4) and (5) we have

$$\begin{aligned} w(\mathcal{A}) &= a_{l-1} + a_l + a_k + a_{k+1} \\ &\le \nabla a_{l-1} + a_l + a_k + \Delta a_{k+1} \le \binom{n-1}{l-1} + \binom{n-1}{k}. \end{aligned}$$

So we have to prove that

$$\binom{n-1}{k} + \binom{n-1}{l-1} < \gamma_d(1,1),$$

i.e., by Lemma 2, it is enough to show that

$$\binom{2d}{d+3} + \binom{2d}{d} < \sum_{j=1}^{d} \left(\binom{2j-1}{j+1} + \binom{2j}{j+1} \right) + 2. \tag{6}$$

It is true for $d = 1$, and when we change d to $d+1$, the increase of the left hand side is

$$\begin{aligned} \mathrm{LHS}(d+1) - \mathrm{LHS}(d) &= \binom{2d+2}{d+4} - \binom{2d}{d+3} + \binom{2d+2}{d+1} - \binom{2d}{d} \\ &= \binom{2d+1}{d+4} + \binom{2d}{d+2} + \binom{2d}{d+1} + \binom{2d+1}{d} \\ &= \binom{2d+1}{d-3} + \binom{2d+1}{d+2} + \binom{2d+1}{d+1}, \end{aligned}$$

which is less than the increase of the right hand side,

$$\begin{aligned} \mathrm{RHS}(d+1) - \mathrm{RHS}(d) &= \binom{2d+1}{d+2} + \binom{2d+2}{d+2} \\ &= \binom{2d+1}{d-1} + \binom{2d+1}{d+2} + \binom{2d+1}{d+1}, \end{aligned}$$

so (6) holds for all d.

Let now $n \geq 2d+2$. We use the weights $w_j = d+2-j$, so $w_d = 2$, $w_{d+1} = 1$. Then

$$w(\mathcal{A}) = (d+2-(l-1))\, a_{l-1} + (d+2-l)\, a_l \\ + (d+2-k)\, a_k + (d+2-(k+1))\, a_{k+1} \leq 2a_{l-1} + a_l,$$

since $d+2-k<0$. But, by (4) and (5),

$$2a_{l-1} + a_l = a_{l-1} + (a_{l-1} + a_l) < \binom{n-1}{l-2} + (\nabla a_{l-1} + a_l) \\ \leq \binom{n-1}{l-2} + \binom{n-1}{l-1} = \binom{n-1}{d-1} + \binom{n-1}{d} = \binom{n}{d},$$

thus it is necessary only to verify that

$$\binom{n}{d} < \gamma_d(2,1). \tag{7}$$

If $n = 3d+1$, it means by Lemma 3 that

$$\binom{3d+1}{d} < 2\sum_{j=2}^{2d+1} \binom{\left\lceil \frac{3}{2}(j-1) \right\rceil - 1}{j} + \sum_{j=2}^{d+1} \binom{3(j-1)}{j} + 3, \tag{8}$$

which is true for $d=1$. Moreover,

$$\mathrm{LHS}(d+1) - \mathrm{LHS}(d) = \binom{3d+4}{d+1} - \binom{3d+1}{d} \\ = \binom{3d+3}{d} + \binom{3d+2}{d} + \binom{3d+1}{d+1},$$

which is easily seen less than

$$\mathrm{RHS}(d+1) - \mathrm{RHS}(d) = 2\binom{3d+2}{d-1} + 2\binom{3d+1}{d-1} + \binom{3d+3}{d+2},$$

hence (8) holds for all d. By (2) in Lemma 3, if $n \geq 3d+1$ then $\gamma_d(2,1)$ is growing faster in n than $\binom{n}{d}$, so (7) is valid for all $n \geq 3d+1$.

Let us now prove (7) for $2d+2 \leq n \leq 3d$. If $n-d$ is odd, then put $n_1 = 3n/2 - 3d/2 - 1/2$, $d_1 = n/2 - d/2 - 1/2$. Denote the right hand side of

(1) in Lemma 3 by $f(n,d)$, and remember that for $n=3d+1$, (1) and (2) in Lemma 3 are the same. Since $n_1=3d_1+1$, $d_1\geq 1$ we have just proven that

$$\binom{n_1}{d_1} < f(n_1,d_1).$$

We observe $n-d=n_1-d_1$, so we need

$$\binom{n}{d}-\binom{n_1}{d_1}\leq f(n,d)-f(n_1,d_1)=\binom{n}{d+1}-\binom{n_1}{d_1+1},$$

that is,

$$\binom{n_1}{n_1-d_1-1}-\binom{n_1}{n_1-d_1}\leq\binom{n}{n-d-1}-\binom{n}{n-d}.$$

It is true, because the function $a\mapsto\binom{a}{t-1}-\binom{a}{t}$, a,t are integers, is increasing for $t\leq a\leq 2t-2$ and $n_1\leq n$. Consequently, (7) is true when $n-d$ is odd. If $n=3$, $d=1$, (7) can be checked directly. In all other cases, if $n-d$ is even, let $n_2=3n/2-3d/2-1$, $d_2=n/2-d/2-1$, so $n_2=3d_2+2$, and a very same argument shows the statement.

3.2. Case 2. $k\geq d+2$, $l\leq d$

We will use the weights $w_j=d+(n-2d-1)(d+1-j)$ in the remaining three sections, so $w_d=n-d-1$, $w_{d+1}=d$. Obviously, $w_l>0$, and remember that $a_{l-1}\neq 0$ implies $l>1$, and $a_{k+1}\neq 0$ implies $k<n-1$. If $w_k>0$, then by (4) and (5),

$$\begin{aligned}
w(\mathcal{A}) &= w_{l-1}a_{l-1}+w_la_l+w_ka_k+w_{k+1}a_{k+1}\\
&= w_l\left(\frac{w_{l-1}}{w_l}a_{l-1}+a_l\right)+w_k\left(a_k+\frac{w_{k+1}}{w_k}a_{k+1}\right)\\
&\leq w_l\left(\frac{n-d-1}{d}a_{l-1}+a_l\right)+w_k\left(a_k+\frac{d}{n-d-1}a_{k+1}\right)\\
&\leq w_l\left(\frac{n-l+1}{l-1}a_{l-1}+a_l\right)+w_k\left(a_k+\frac{k+1}{n-k-1}a_{k+1}\right)\\
&\leq w_l(\nabla a_{l-1}+a_l)+w_k(a_k+\Delta a_{k+1})\\
&\leq w_l\binom{n-1}{l-1}+w_k\binom{n-1}{k}. \qquad (9)
\end{aligned}$$

It is easy to see that for all $0<i\le d$, and $d\le j<n-1$, $w_j>0$ we have

$$w_i\binom{n-1}{i-1}<w_{i+1}\binom{n-1}{i},\qquad w_{j+1}\binom{n-1}{j+1}<w_j\binom{n-1}{j},\tag{10}$$

since

$$\frac{w_i}{w_{i+1}}\le\frac{n-d-1}{d}<\frac{n-i}{i},\qquad \frac{w_{j+1}}{w_j}\le\frac{d}{n-d-1}<\frac{j+1}{n-j-1}.$$

Thus, for $k\ge d+2$, $l\le d$ and $w_k>0$ by (9), (10) it holds

$$\begin{aligned}w(\mathcal{A})&\le w_d\binom{n-1}{d-1}+w_{d+2}\binom{n-1}{d+2}\\&=(n-d-1)\binom{n-1}{d-1}+(-n+3d+1)\binom{n-1}{d+2}\\&\le(n-d-1)\binom{n-1}{d-1}+d\,\frac{d-1}{d+2}\binom{n-1}{d+1}.\end{aligned}$$

Though we need $w_k>0$ to prove (9), if $w_k\le 0$ (so $w_{k+1}<0$), trivially

$$w_ka_k+w_{k+1}a_{k+1}\le d\,\frac{d-1}{d+2}\binom{n-1}{d+1}.$$

So in order to show

$$w(\mathcal{A})<\gamma_d(n-d-1,d),$$

it is enough to prove that

$$(n-d-1)\binom{n-1}{d-1}+d\,\frac{d-1}{d+2}\binom{n-1}{d+1}<\gamma_d(n-d-1,d).\tag{11}$$

Introducing the notation

$$\mu_i=\left\lfloor\frac{d-i}{d}(n-1)\right\rfloor,$$

by Lemma 4, (11) can be written as

$$(n-d-1)\sum_{i=1}^{d-1}\binom{\mu_i}{d-i}<\frac{3d}{d+2}\binom{n-1}{d+1}+d\sum_{i=1}^{d-1}\binom{\mu_i}{d-i+1}+d.$$

It is true for $d=1,2$, and since

$$\binom{n-1}{d+1} = \sum_{i=1}^{d+2} \binom{n-i-1}{d-i+2},$$

it is enough to verify

$$(n-d-1)\binom{\mu_i}{d-i} \le \frac{3d}{d+2}\binom{n-i-1}{d-i+2} + d\binom{\mu_i}{d-i+1},$$

if $d \ge 3$, $1 \le i \le d-1$. Let us devide both sides by $\binom{\mu_i}{d-i}$. Note that from $n \ge 2d+1$ it follows $n-i-1 \ge \mu_i + 1$ for all i, moreover, if $i \ge 2$ then $n-i-1 \ge \mu_i + 2$ holds, too. If $n-i-1 \ge \mu_i + 2$, we obtain

$$\begin{aligned} n-d-1 \le {} & \frac{3d}{d+2} \cdot \frac{(n-i-1)(n-i-2)(n-i-3)\cdots(\mu_i+1)}{(d-i+1)(d-i+2)(n-d-3)\cdots(\mu_i-d+i+1)} \\ & + d\,\frac{\mu_i-d+i}{d-i+1}. \end{aligned} \tag{12}$$

Since $n-i-2 \ge d-i+2$, $n-i-3 > n-d-3, \ldots, \mu_i+1 > \mu_i-d+i+1$ we will prove only

$$n-d-1 \le \frac{3d}{d+2} \cdot \frac{n-i-1}{d-i+1} + d\,\frac{\mu_i-d+i}{d-i+1}.$$

We observe

$$\begin{aligned} d(\mu_i-d+i) &= d\left(\left\lfloor \tfrac{d-i}{d}(n-1)\right\rfloor - (d-i)\right) \ge d\left(\tfrac{d-i}{d}(n-1) - \tfrac{d-1}{d} - (d-i)\right) \\ &= d\left(\tfrac{d-i}{d}(n-d-1) - \tfrac{d-1}{d}\right) = (d-i)(n-d-1) - (d-1), \end{aligned}$$

hence

$$n-d-1-d\,\frac{\mu_i-d+i}{d-i+1} \le \frac{n-2}{d-i+1}, \tag{13}$$

so we need to verify

$$\frac{n-2}{d-i+1} \le \frac{3d}{d+2} \cdot \frac{n-i-1}{d-i+1}.$$

After multiplying both sides by $d-i+1$, we put $i=d-1$ since the right hand side is decreasing in i, so we get

$$(n-2)(d+2) \le 3d(n-d),$$

which is true for $n \ge 2d+1$.

Let us show now the missing case $n-i-1=\mu_i+1$, so $i=1$. Instead of (12) we have

$$n-d-1 \le \frac{3d}{d+2}\cdot\frac{(n-2)(n-d-2)}{(d+1)d}+d\,\frac{n-d-2}{d},$$

that is,

$$(d+2)(d+1) \le 3(n-2)(n-d-2),$$

which is true for $d\geq 3$, $n\geq 2d+1$. Thus, we have proven (11).

3.3. Case 3. $k=d+1$, $l\leq d-1$

If $w_k>0$, by (9) and (10),

$$w(\mathcal{A}) \le w_{d-1}\binom{n-1}{d-2}+w_{d+1}\binom{n-1}{d+1}.$$

Notice that if $w_k\leq 0$, then

$$w_k a_k + w_{k+1}a_{k+1} < d\binom{n-1}{d+1}$$

holds, so we want to prove

$$(2n-3d-2)\binom{n-1}{d-2}+d\binom{n-1}{d+1} < \gamma_d(n-d-1,d).$$

But

$$(2n-3d-2)\binom{n-1}{d-2} \le (n-d-1)\frac{d-1}{d+1}\binom{n-1}{d-1},$$

thus, by Lemma 4, we show

$$(n-d-1)\sum_{i=1}^{d-1}\binom{\mu_i}{d-i} < \frac{2(n-d-1)}{d+1}\binom{n-1}{d-1}+d\sum_{i=1}^{d-1}\binom{\mu_i}{d-i+1}+d.$$

Since

$$\binom{n-1}{d-1}=\sum_{i=1}^{d}\binom{n-i-1}{d-i},$$

it is necessary only to prove

$$(n-d-1)\binom{\mu_i}{d-i} \le \frac{2(n-d-1)}{d+1}\binom{n-i-1}{d-i}+d\binom{\mu_i}{d-i+1}$$

for all $1 \le i \le d-1$. Remember that $n-i-1 \ge \mu_i+1$ so after dividing by $\binom{\mu_i}{d-i}$ we get

$$n-d-1 \le \frac{2(n-d-1)}{d+1} \cdot \frac{(n-i-1)\cdots(\mu_i+1)}{(n-d-1)\cdots(\mu_i-d+i+1)} + d\,\frac{\mu_i-d+i}{d-i+1}.$$

Since $n-d-1 \le n-i-2, \ldots, \mu_i-d+i+2 \le \mu_i+1$, similarly to Case 2, by (13), we need to verify

$$\frac{n-2}{d-i+1} \le \frac{2(n-d-1)}{d+1} \cdot \frac{n-i-1}{\mu_i-d+i+1}.$$

We have $n-2 \le 2(n-d-1)$, so the only thing to prove is

$$d+1 \le \frac{(n-i-1)(d-i+1)}{\mu_i-d+i+1}, \tag{14}$$

or

$$(d+1)\left(\tfrac{d-i}{d}(n-1)-d+i+1\right) \le (n-i-1)(d-i+1). \tag{15}$$

It is equivalent to

$$0 \le \tfrac{i}{d}n-(i+1)(d-i+1)+\tfrac{d-i}{d}(d+1)+(d-i-1)(d+1),$$

which is increasing in n, so it is enough to check (15) for $n=2d+1$, and we have

$$(d+1)(d-i+1) \le (2d-i)(d-i+1),$$

which is valid, because $i \le d-1$.

3.4. Case 4. $k=d$, $l \le d-2$

We prove the case $3d \le n$ directly. So an antichain $\mathcal{A}=\mathcal{A}_{l-1}\cup\mathcal{A}_l\cup\mathcal{A}_d\cup\mathcal{A}_{d+1}$ is given and $\mathcal{B}=\mathcal{B}_d\cup\mathcal{B}_{d+1}$ is its flat counterpart. To have equal size and volume, it is easy to see that

$$b_d = a_d + a_{l-1} + a_l + z,$$
$$b_{d+1} = a_{d+1} - z,$$

where $z=(d-l)a_l+(d-l+1)a_{l-1}$. Since $\mathcal{A}_d\cup\mathcal{A}_{d+1}\subseteq 2^{[n-1]}$, in order to show that $\mathcal{B}$ can be an antichain, it is enough to find $a_{l-1}+a_l+z$ d-element sets that contain n, so we have to prove

$$a_{l-1}+a_l+z \le \binom{n-1}{d-1},$$

or equivalently,

$$(d-l+2)a_{l-1}+(d-l+1)a_l \le \binom{n-1}{d-1}.$$

If $l>1$, by (4) and (5)

$$\begin{aligned}(d-l+2)a_{l-1}+(d-l+1)a_l &= (d-l+1)\left(\frac{d-l+2}{d-l+1}\,a_{l-1}+a_l\right)\\ &\le (d-l+1)\left(\frac{n-l+1}{l-1}\,a_{l-1}+a_l\right)\\ &\le (d-l+1)(\nabla a_{l-1}+a_l)\\ &\le (d-l+1)\binom{n-1}{l-1}.\end{aligned}$$

If $l=1$ then $a_{l-1}=0$ and $a_l\le\binom{n-1}{l-1}$. Consequently, we need to show

$$(d-l+1)\binom{n-1}{l-1} \le \binom{n-1}{d-1},$$

which is true, because

$$d-l+1 \le 2^{d-l} \le \frac{(n-d+1)\cdots(n-l)}{(d-1)\cdots l},$$

for $3d\le n$.

Let us finally study the case $2d+1\le n\le 3d$, and note that $d\ge l+2$ implies $d\ge 3$. Obviously, if $w_k\le 0$, then

$$w_k a_k + w_{k+1}a_{k+1} < (n-d-1)\binom{n-1}{d}.$$

Otherwise, by (9) and (10),

$$w(\mathcal{A}) \le w_{d-2}\binom{n-1}{d-3} + w_d\binom{n-1}{d},$$

hence it is necessary only to show that

$$(3n-5d-3)\binom{n-1}{d-3} + (n-d-1)\binom{n-1}{d} < \gamma_d(n-d-1,d).$$

By Lemma 4, it is enough to prove

$$(n-d-1)\sum_{i=1}^{d-1}\binom{\mu_i}{d-i} < g(n,d)\binom{n-1}{d-1} + d\sum_{i=1}^{d-1}\binom{\mu_i}{d-i+1} + d, \tag{16}$$

where

$$g(n,d) = (n-d-1)\frac{d-1}{d+1} - \frac{(3n-5d-3)(d-1)(d-2)}{(n-d+1)(n-d+2)},$$

because

$$\binom{n-1}{d-3} = \frac{(d-1)(d-2)}{(n-d+1)(n-d+2)}\binom{n-1}{d-1},$$

$$(n-d-1)\binom{n-1}{d} = (d+1)\binom{n-1}{d+1},$$

and

$$\begin{aligned}(n-d-1)\binom{n-1}{d-1} - \binom{n-1}{d+1} &= (n-d-1)\left(1-\frac{n-d}{d(d+1)}\right)\binom{n-1}{d-1}\\ &\geq (n-d-1)\frac{d-1}{d+1}\binom{n-1}{d-1}\end{aligned}$$

for $n \leq 3d$. For $n \geq 2d+1$, $g(n,d) > 0$ holds since

$$g(2d+1,d) = \frac{d(d-1)(6d+8)}{(d+1)(d+2)(d+3)} > 0,$$

and

$$\begin{aligned}g(n+1,d)-g(n,d) &= \frac{d-1}{d+1} + \frac{(3n-7d-9)(d-1)(d-2)}{(n-d+1)(n-d+2)(n-d+3)}\\ &\geq \frac{1}{d+1}.\end{aligned} \tag{17}$$

To see this latter inequality, we claim that

$$(n-d+1)(n-d+2)(n-d+3) \geq -(3n-7d-9)(d-1)(d+1).$$

If the right hand side is negative, then it is clearly true; otherwise, since $n-d+1 \geq d+1$, it is enough to show

$$(n-d+2)(n-d+3) \geq -(3n-7d-9)(d-1). \tag{18}$$

It is equivalent to

$$(n-d)^2+(3d+2)(n-d)-(4d+9)(d-1)+6\geq 0,$$

and the left hand size is increasing in $n-d$, so we need to check (18) for $n=2d+1$, and we get

$$(d+3)(d+4)\geq (d+6)(d-1),$$

which is valid. So $g(n,d)$ is really non-negative, thus instead of (16) we prove

$$(n-d-1)\binom{\mu_i}{d-i}\leq g(n,d)\binom{n-i-1}{d-i}+d\binom{\mu_i}{d-i+1}.$$

As in the previous cases, we devide both sides by $\binom{\mu_i}{d-i}$ and, by (13), (14), we obtain to show

$$n-2\leq g(n,d)(d+1).$$

But (17) assures that the right hand size is growing in n at least as fast as the left hand side, so we verify the inequality only for $n=2d+1$. We have

$$2d-1\leq \frac{d(d-1)(6d+8)}{(d+2)(d+3)},$$

which is true for $d=3$, while for $d\geq 4$ we know that

$$2\leq \frac{(d-1)(6d+8)}{(d+2)(d+3)},$$

and we are done.

Thus, we have proven all the possible cases, and so the Flat Antichain Conjecture.

Acknowledgement. The author is indebted to A. Sali for drawing his attention to the problem.

References

[1] L. Brankovic, P. Lieby and M. Miller: Flattening Antichains with Respect to the Volume, *Electronic J. Combs.* **6** (1999).

[2] G. F. Clements: A minimization problem concerning subsets of a finite set, *Disc. Math.* **4** (1973), 123–128.

[3] D. E. Daykin, J. Godfrey and A. J. W. Hilton: Existence Theorems for Sperner Families, *J. Comb. Theory Ser. A* **17** (1974), 245–251.

[4] Á. Kisvölcsey: Profiles of 2-level non-Sperner families, *Studia Sci. Math. Hung.* **39** (2002), 189–202.

[5] D. J. Kleitman and E. C. Milner: On the Average Size of the Sets in a Sperner Family, *Disc. Math.* **6** (1973), 141–147.

[6] P. Lieby: *The Separation Problem*, Honours thesis, Northern Territory University, 1994.

[7] P. Lieby: *Extremal Problems in Finite Sets*, PhD. thesis, Northern Territory University, 1999.

[8] P. Lieby: Antichains on Three Levels, *Electronic J. Combs.* **11(1)** (2004), R50.

[9] C. Ramsay and I. T. Roberts: Minimal Completely Separating Systems of Sets, *Australasian J. Combs.* **13** (1996), 129–150.

[10] C. Ramsay, I. T. Roberts and F. Ruskey: Completely Separating Systems of k-Sets, *Disc. Math.* **183** (1998), 265–275.

[11] I. T. Roberts: *The Flat Antichain Conjecture for Small Average Set Size*, 1999.

[12] E. Sperner: Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27** (1928), 544–548.

Ákos Kisvölcsey

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
1053 Budapest, Reáltanoda u. 13–15.
Hungary
ksvlcs@renyi.hu